Question Sheet 5, Continuity II

v1 2020-21

Intermediate Value Theorem continued

1. Assume that $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists some $c \in [0, 1]$ such that

$$f(c) = \sin\left(\frac{\pi c}{2}\right).$$

Hint Apply the Intermediate Value Theorem to

$$h(x) = f(x) - \sin\left(\frac{\pi x}{2}\right).$$

2. Prove a version of the *Fixed Point Theorem.* If $f, g : [a, b] \to [a, b]$ are continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$ then there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Hint: Follow the hint in the previous question and consider $h(x) = f(x) - g(x)$.

3. Prove that if f is a bounded continuous function on $\mathbb{R}$ then there exists $c \in \mathbb{R}$ such that $f(c) = c^3$.

Boundedness Theorem

4. Recall the **Boundedness Theorem** which states that a *continuous function on a closed bounded interval is bounded and attains its bounds.* In this question we check if the conditions that the function be *continuous* on a *closed, bounded interval* are necessary. So, if remove any of these conditions does the conclusion of the Theorem still hold?

Give examples of

i) A function on a closed bounded interval that is not bounded.

ii) A continuous function on $(-1, 1)$ with range $(-\infty, \infty)$, (and thus is not bounded).

iii) A function on $[0, 1]$ that is bounded but does not attain its bounds.

5. i) a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x^2+1}$$

is bounded for all $x \in \mathbb{R}$.

b) Does f attain its bounds?

c) Is this a counter-example to the Boundedness Theorem, in particular that functions continuous on a closed bounded interval attain their bounds?

ii) a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{x^2+1}$$

is bounded for all $x \in \mathbb{R}$.

b) Does f attain its bounds?

iii) Sketch the graphs of both functions.

Hint: Expand and rearrange the inequalities

$$(x-1)^2 \geq 0 \quad \text{and} \quad (x+1)^2 \geq 0.$$

Strictly Monotonic functions

6. Prove that

i) for all $n \in \mathbb{N}$, with n even, then x^n is strictly increasing on $[0, \infty)$,

ii) for all $n \in \mathbb{N}$, with n even, then x^n is **not** strictly increasing on $\mathbb{R}$,

iii) for all $n \in \mathbb{N}$, with n odd, then x^n is strictly increasing on $\mathbb{R}$.

Hint: use the factorization

$$x^n - y^n = (x-y)\left(x^{n-1} + yx^{n-2} + y^2x^{n-3} + \ldots + y^{n-2}x + y^{n-1}\right).$$

In (iii) it might help to look at 3 cases, $x > y \geq 0$, $x > 0 > y$ and $0 \geq x > y$.

7. Prove that the hyperbolic functions $\sinh x$ and $\tanh x$ are strictly increasing on $\mathbb{R}$ while $\cosh x$ is strictly increasing on $[0, \infty)$.

   Hint. Prove that
   $$\sinh(x+y) > \sinh x$$
   for all $x \in \mathbb{R}$ and $y > 0$, similarly for tanh, while
   $$\cosh(x+y) > \cosh x$$
   for all $x, y > 0$.

Inverse Function Theorem

8. State the Inverse Function Theorem.

   Explain how to define the following inverse functions,

   i) $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$,

   ii) $\cosh^{-1} : [1, \infty) \to [0, \infty)$.

   iii) $\tanh^{-1} : (-1, 1) \to \mathbb{R}$.

   Don't forget to show that the inverses map between the sets shown.

   A problem might be that the Inverse Function Theorem as stated in lectures refers to bounded interval while here we have $\mathbb{R}$, $[1, \infty)$ and $[0, \infty)$. An approach might be to take a large N and consider sinh and tanh on $[-N, N]$ and cosh on $[0, N]$, define their inverses and finish by letting $N \to \infty$.

Logarithm

9. Prove that the natural logarithm, defined as the inverse of the exponential function, satisfies
   $$\ln a + \ln b = \ln ab$$
   for all $a, b > 0$.

   (As throughout this course you may assume that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$.)

   Hint What are $e^{\ln a + \ln b}$ and $e^{\ln ab}$? You may need to use the fact that e^x is an injective function.

Additional Questions for practice

10. Show that

$$2\cos^2 x + 3\cos x + 1 = 2x^2 + 3x + 1$$

has **a** solution in $[0, \pi/2]$.

11. Show that

$$\frac{x}{\sin x} + \frac{1}{\cos x} = \pi$$

has **a** solution with $x \in (0, \pi/2)$.